**MTH 2310, FALL 2011**

TEST 2 REVIEW, KEY

Some problems to work on in class today (most of these are even-numbered problems from the textbook):

(1) True/False: If True, justify your answer with a brief explanation. If False, give a counterexample or a brief explanation.

(a) If $A$ and $B$ are $3 \times 3$ and $B = \begin{bmatrix} \mathbf{b}_1 & \mathbf{b}_2 & \mathbf{b}_3 \end{bmatrix}$, then $AB = \begin{bmatrix} A\mathbf{b}_1 + A\mathbf{b}_2 + A\mathbf{b}_3 \end{bmatrix}$. **False**, should not have plusses.

(b) If $A$ is invertible, then the inverse of $A^{-1}$ is $A$ itself. **True**. $(A^{-1})^{-1} = A$ is a property of inverses.

(c) Let $A$ be a square matrix. If the equation $A\mathbf{x} = \mathbf{b}$ has at least one solution for each $\mathbf{b}$ in $\mathbb{R}^n$, then the solution is unique for each $\mathbf{b}$. **True**. Because of at least one solution we know it is onto, and by the IMT this means it is one-to-one.

(d) If $A_1$, $A_2$, $B_1$ and $B_2$ are $n \times n$ matrices, $A = \begin{bmatrix} A_1 \\ A_2 \end{bmatrix}$ and $B = \begin{bmatrix} B_1 & B_2 \end{bmatrix}$, then the product $BA$ is defined but $AB$ is not. **False**, both are defined.

(e) The determinant of a triangular matrix is the sum of the entries on the diagonal. **False**, should be product, not sum.

(f) If $\det A = 0$, then two rows or two columns are the same, or a row or column is zero. **False**, could have one row be a sum of two other rows.

(g) A vector space is also a subspace. **True**, it is a subspace of itself.

(h) The column space Col $A$ is not affected by elementary row operations on $A$. **False**.

(2) Show that if the columns of $B$ are linearly independent, then so are the columns of $AB$. (Hint: First, explain why the columns of $B$ being linearly dependent means the same thing as saying that there is a nonzero vector $\mathbf{v}$ such that $B\mathbf{v} = \mathbf{0}$. Now use this to answer the question.)

This is not true, because $A$ could be the zero matrix. The question should read "columns of $B$ are linearly *de*pendent", and then it follows from the hint.

(3) Find the inverse of $\begin{bmatrix} 1 & -2 & 1 \\ 4 & -7 & 3 \\ -2 & 6 & -4 \end{bmatrix}$.

Put this into a online calculator to check your work. You should do it by hand using the algorithm where you augment it with the identity matrix and then row reduce.

(4) If $L$ is an $n \times n$ matrix and the equation $L\mathbf{x} = \mathbf{0}$ has the trivial solution, do the columns of $L$ span $\mathbb{R}^n$? Why or why not?

Not necessarily. If the equation has *only* the trivial solution, then it will span (by the IMT). But all such equations have the trivial solution, and some of them have more.

(5) Let $A = \begin{bmatrix} B & 0 \\ 0 & C \end{bmatrix}$, where $B$ and $C$ are square. Prove that if $A$ is invertible then $B$ and $C$ must be invertible. (Note: you cannot just invoke the Invertible matrix Theorem, you have to use block matrix multiplication at some point.)

$A$ invertible means there is some block matrix $\begin{bmatrix} X & Y \\ W & Z \end{bmatrix}$ such that $\begin{bmatrix} B & 0 \\ 0 & C \end{bmatrix} \begin{bmatrix} X & Y \\ W & Z \end{bmatrix} = I$. This means that $BX = I$ so $B$ must be invertible, etc.

(6) Calculate $\det \begin{bmatrix} 6 & 3 & 2 & 4 & 0 \\ 9 & 0 & -4 & 1 & 0 \\ 8 & -5 & 6 & 7 & 1 \\ 3 & 0 & 0 & 0 & 0 \\ 4 & 2 & 3 & 2 & 0 \end{bmatrix}$.

9.

(7) Use row reduction to find $\det \begin{bmatrix} 1 & 3 & 3 & -4 \\ 0 & 1 & 2 & -5 \\ 2 & 5 & 4 & -3 \\ -3 & -7 & -5 & 2 \end{bmatrix}$.

0.

(8) Let $U$ be a square matrix such that $U^T U = I$. Show that either $\det U = 1$ or $\det U = -1$.

$\det(U^T U) = \det(I)$ implies $\det(U^T) \cdot \det(U) = 1$. But $\det U = \det U^T$, so they have to be $\pm 1$.

(9) Let $W$ be the set of all vectors of the form $\begin{bmatrix} 4a + 3b \\ 0 \\ a + b + c \\ c - 2a \end{bmatrix}$. Is $W$ a subspace of $\mathbb{R}^4$? Explain your reasoning.

Yes it is a subspace because it is the column space of the matrix $\begin{bmatrix} 4 & 3 & 0 \\ 0 & 0 & 0 \\ 1 & 1 & 1 \\ -2 & 0 & 1 \end{bmatrix}$.

(10) Consider the following two systems of equations:

$$\begin{array}{rrrl} 5x_1+ & x_2- & 3x_3 = & 0 \\ -9x_1+ & 2x_2+ & 5x_3 = & 1 \\ 4x_1+ & x_2- & 6x_3 = & 9 \end{array} \quad \text{and} \quad \begin{array}{rrrl} 5x_1+ & x_2- & 3x_3 = & 0 \\ -9x_1+ & 2x_2+ & 5x_3 = & 5 \\ 4x_1+ & x_2- & 6x_3 = & 45 \end{array}$$

It can be shown that the first system has a solution. Use this fact to explain why the second system also has a solution without making any row operations.

Ans: The fact that there is a solution means there exists some $\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$ such that $\begin{bmatrix} 5 & 1 & 3 \\ -9 & 2 & 5 \\ 4 & 1 & 6 \end{bmatrix} \mathbf{x} = \begin{bmatrix} 0 \\ 1 \\ 9 \end{bmatrix}$. Then

$$\begin{bmatrix} 5 & 1 & 3 \\ -9 & 2 & 5 \\ 4 & 1 & 6 \end{bmatrix} (5\mathbf{x}) = 5 \begin{bmatrix} 5 & 1 & 3 \\ -9 & 2 & 5 \\ 4 & 1 & 6 \end{bmatrix} \mathbf{x} = 5 \begin{bmatrix} 0 \\ 1 \\ 9 \end{bmatrix} = \begin{bmatrix} 0 \\ 5 \\ 45 \end{bmatrix}$$

which is a solution to the second system.